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DIFFERENTIATION BY 'CALCULATING-SOLVING' INSTRUMENTS

I. M. Rapoport
Submitted by Academician
N. G. Bruevich

Izv. Ak. Nauk SSSR, Ot Tekh. Nauk, No 11 (Nov 1947), pp 1521-1542.

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Introduction

From design of tachometers can be distinguished two categories: tachometers that determine the average value of a measurable velocity over a known interval of time providing the obtaining of the result of measurement; and tachometers that determine with sufficient accuracy the instantaneous value of measurable velocity. Examples of tachometers of the first category are the widely-used tachometers that find the increment of the differentiable function for a certain "observation time" T and determine in this manner the average arithmetical value of the measurable velocity over this time T . An example of a tachometer of the second category is the electrical tachometer, an electrical machine that gives the voltage proportional to the velocity of rotation of its rotor and determines thus the angular velocity of rotation of the axle rod connected with the rotor of the machine. Tachometers that determine the average value of the velocity are employed only for measuring constant speeds, but even with this deficiency they still possess great value. For a sufficiently large time T of observation, they can guarantee high accuracy in measurements of constant velocity, even if the differentiable function enters the tachometer with considerable errors; in other words, these tachometers can always secure sufficient insensitivity to errors in the input data because of the large observable time. Tachometers that determine with sufficient accuracy the instantaneous value of the measurable velocity are employed also for measuring variable velocities; in this connection they suffer a great defect. If the differentiable function of time $x(t)$ enters the tachometer with an error $\delta x(t)$, then in determining the derivative dx/dt we obtain an error equal to $d\delta x/dt$. Thus, we see, by guarantee-

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only the known accuracy of the input function $x(t)$ ($|\delta x(t)| \leq \epsilon$) into the calculation we do cannot guarantee any definite accuracy of measurement by the tabulation of the derivative, namely dx/dt .

¶ As a consequence of this property of tabulation, we avoid employing, in differentiating input units, calculating instruments [Note: calculating-instruments] that require high accuracy and smooth determination of the output.

¶ Thus the problem of accuracy of differentiation, which is an open question, would guarantee a definite accuracy of finding the desired derivatives if the limits of variation of the input data can be assigned and, on the other hand, could be employed for tabulating a definite class of functions in the class of functions of time; that is, we can make a definite statement about the class of functions.

¶ The present article is devoted to the problem of differentiating in calculating instruments the derivatives dx/dt , d^2x/dt^2 , ..., d^nx/dt^n under the condition that the differentiable function of time $x(t)$ satisfy a given differential equation:

$$F(d^{n+1}x/dt^{n+1}, d^2x/dt^2, \dots, dx/dt, x) = 0. \quad (1)$$

(Note: The argument with respect to which the differentiation is being carried out shall be considered the time t everywhere in this work. In the general case the derivative dy/dx can always be interpreted in a calculating-solving instrument as the ratio of two speeds $dy/dt : dx/dt$. We shall assume that time t does not explicitly enter the differential equation. In the general case one can exclude from the equation the time t explicitly appearing in it by increasing the order of the equation by unity.)

¶ As we shall show below, by limiting the class of functions subject to differentiation to differential equations which must be satisfied by the function to be differentiated, we can guarantee a definite accuracy of finding the

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the desired derivatives only under the condition that one indicate the limits of errors of measurements introduced by the function under study into the differentiating unit, or terminal, of the calculating-solving instrument.

¶ This article is basically devoted to the mathematical side of the problem; in addition to this, several examples are also shown of differentiating units of both mechanical and electrical calculating-solving instruments.

¶ Paragraph 1. Let us dwell at length on the case where the function of time $x(t)$, whose derivative is subjected to measurements, varies with constant velocity; that is:

$$\frac{dx}{dt} = \text{const.} \quad (2)$$

Let us consider the function of time:

$$y(t) = \int_0^{\infty} \frac{dx}{dt}(t-\tau) p(\tau) d\tau \quad (3)$$

where $p(\tau)$ is a function satisfying the condition:

$$\int_0^{\infty} p(\tau) d\tau = 1. \quad (p(\tau) \equiv \text{weight function}) \quad (4)$$

In the general case the function of time $y(t)$ is the average value of the derivative dx/dt over the time preceding the current moment of time t calculated (that is, the derivative) with the weight $p(\tau)$. In the particular case where the derivative dx/dt maintains a constant value, we have:

$$y(t) = \frac{dx}{dt} \int_0^{\infty} p(\tau) d\tau = \frac{dx}{dt}; \quad (4')$$

that is, the function $y(t)$ maintains a constant value equal to the derivative dx/dt . If up to the moment $t=0$ the derivative dx/dt equals zero, but the derivative dx/dt starting at this moment of time maintains a certain constant value, then we have:

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$$\left. \begin{aligned} y(t) &= 0 \text{ for } t \leq 0 \\ y(t) &= \frac{dx}{dt} \int_0^t p(\tau) d\tau \text{ for } t \geq 0. \end{aligned} \right\} \quad (5)$$

4. In the case for $t \geq 0$ the function $y(t)$ differs from the derivative dx/dt and tends towards it as time t increases without limit. Thus, if one determines for the derivative dx/dt the value of $y(t)$ according to Formula (3) and begins the measurement from the moment of time $t=0$, then we shall obtain the magnitude of the derivative dx/dt with the relative error:

$$\vartheta(t) = 1 - \int_0^t p(\tau) d\tau = \int_t^\infty p(\tau) d\tau. \quad (6)$$

If the function $x(t)$ is defined with an error $\delta x(t)$, then seeking for derivative dx/dt , the value $y(t)$ in accordance with Formula (3), we obtain the error:

$$\Delta(t) = \int_0^t \frac{d\delta x}{dt} (t-\tau) p(\tau) d\tau = \delta x(t) p(0) + \int_0^t \delta x(t-\tau) p(\tau) d\tau. \quad (6')$$

$$1) \quad |\delta x(\tau)| \leq \varepsilon \text{ for } \tau \leq t,$$

$$\text{then } |\Delta(t)| \leq \varepsilon \left[|p(0)| + \int_0^t |p(\tau)| d\tau \right]. \quad (7)$$

$$2) \quad p(\tau) = 0 \text{ for } \tau > T, \quad (8)$$

then Formula (3) assumes the form:

$$y(t) = \int_0^T \frac{dx}{dt} (t-\tau) p(\tau) d\tau. \quad (9)$$

In this case the function of time $y(t)$ is the average value of the derivative dx/dt over time from the moment of time $t-T$ to the moment t or, as they say, over the observation time T calculated with weight $p(\tau)$. Therefore:

$$\vartheta(t) = 0 \text{ for } t \geq T \quad (9')$$

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that is, by beginning, the measurement of the derivative dx/dt from the moment of time $t=0$, we shall have, after a lapse of time T of observation, the exact value of this derivative dx/dt .

If the function $x(t)$ is determined within error $\delta x(t)$, then in seeking, for the derivative dx/dt , the value $y(t)$, according with Formula (9), we shall obtain for t greater than T in measurements of the derivative dx/dt the following error:

$$\Delta(t) = \int_0^T \delta x(t-\tau) p(\tau) d\tau = x(t)p(t) - \delta x(t-T)p(T) + \dots \quad (10)$$

$$+ \int_0^T \delta x(t-\tau) p(\tau) d\tau.$$

If $|\delta x(\tau)| \leq \varepsilon$ for $t-T \leq \tau \leq t$,

$$|\Delta(t)| \leq \varepsilon [p(t) + p(T) + \int_0^T p(\tau) d\tau]. \quad (11)$$

If the weight $p(\tau)$ for $0 < \tau < T$ maintains a constant value and equals 0 for $\tau > T$, that is:

$$\left. \begin{aligned} p(\tau) &= \frac{1}{T} & \text{for } 0 < \tau < T, \\ p(\tau) &= 0 & \tau > T. \end{aligned} \right\} \quad (12)$$

then Formula (1) assumes the form:

$$y(t) = \frac{1}{T} \int_0^T \frac{dx}{dt} (t-\tau) d\tau = \frac{x(t) - x(t-T)}{T}. \quad (13)$$

and we present to a determination of the average arithmetic value of the derivative dx/dt over the observation time T , (concerning) which we have already mentioned at the beginning of this article. In this case we have:

$$|\Delta(t)| \leq \frac{2\varepsilon}{T}. \quad (14)$$

if $|\delta x(\tau)| \leq \varepsilon$ for $t-T \leq \tau \leq t$.

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¶ Let us now dwell in greater detail on a method of measuring constant velocity which we shall later generalize.

In order to find the derivative dx/dt , we shall effect the solution of the following differential equation:

$$\frac{dx}{dt} + k(x - x_1) = 0, \quad k > 0 \quad (15)$$

with the initial condition

$$x_1 = x \quad \text{for } t = t_0 \quad (16)$$

and for the desired derivative dx/dt we shall determine the value:

$$y = k(x - x_1). \quad (17)$$

Substituting x_1 from (17) into (15), we obtain for y the differential equation:

$$\frac{dy}{dt} + ky = k \frac{dx}{dt} \quad (18)$$

with the initial conditions:

$$y = 0 \quad \text{for } t = t_0. \quad (19)$$

Solving the differential equation (18) for the initial condition in (19), we find:

$$y(t) = k \int_{t_0}^t \frac{dx}{dt}(s) e^{-k(t-s)} ds = k \int_0^{t-t_0} \frac{dx}{dt}(t-\tau) e^{-k\tau} d\tau. \quad (20)$$

¶ Assuming, that the velocity dx/dt is measured continuously as time t increases without ^{limit} and setting in (20) $t_0 = -\infty$, we are led to the following formula:

$$y(t) = \int_0^{\infty} \frac{dx}{dt}(t-\tau) p(\tau) d\tau. \quad (21)$$

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where $p(\tau) = k e^{-k\tau}$. (22)

Substituting (22) in (6) and (7), we find:

$$D(t) = e^{-kt} \quad (23)$$

$$|\Delta(t)| \leq \epsilon \left(k + \int_0^{\infty} k^2 e^{-k\tau} d\tau \right) = 2k\epsilon. \quad (24)$$

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Formulas (24) and (14) show that the method considered by us for measuring the derivative dx/dt guarantees the same accuracy as in the determination of the average arithmetic value of the derivative over the observation time $T = \frac{1}{k}$. However, as seen from Formula (23), beginning the measurement at the moment of time $t=0$, we still do not receive at the moment of time $t = \frac{1}{k}$ an accurate value of the derivative, but only a value with a relative error of $1/e$.

The differentiating mechanism that we require for a constant speed of dx/dt its average value of $\dot{x}$ in accordance with Formulas (12) and (15) are well known under the name of automatic friction. The principle of this mechanism is shown schematically in Figure 1.

The quantity x which is being subjected to differentiation enters the mechanism in the form of the angular displacement ϕ . The output of the mechanism is the angular displacement ψ which determines for a fixed scale the desired angular velocity $d\phi/dt$:

$$\psi = \lambda \cdot d\phi/dt \quad (25)$$

where lambda λ is the given scale coefficient having the dimensions of time. If we designate by ϕ_1 the angular displacement of the friction shaft, and by γ the angle of turn of the screw controlling the friction carriage ($\gamma=0$ when the carriage assumes the central position), then we have, as seen from Figure 1:

$$\gamma = i(\phi - \phi_1) \quad (26)$$

$$\psi = i_2(\phi - \phi_1), \quad (27)$$

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where i_1 and i_2 are the transmission numbers, whose calculation is conducted below. The origin for angle φ_1 will be taken so that $\varphi_1 = \varphi$ for $\gamma = 0$. The angular velocity of rotation of the friction carriage $d\varphi_1/dt$ is proportional to the angle γ of turn of the screw controlling the friction carriage:

$$\gamma = \tau \cdot d\varphi/dt \quad (28)$$

where $\tau = \pi d / \omega^s$.

It

Here d is the diameter of the friction carriage; s is the pitch of the screw controlling the friction carriage; ω is the angular velocity of rotation of the friction shaft. As is evident from Relation (28), the turn of the friction shaft in the time from t_0 to $t - \varphi_1(t) - \varphi_1(t_0)$ equals the integral:

$$\frac{1}{\tau} \int_{t_0}^t \gamma(t) dt;$$

that is, the friction mechanism by itself is an integrating member. Conversion of the friction mechanism, during which the mechanism remains a mechanism carrying out the operation of differentiation, is achieved by external friction represented in Figure 1. Substituting (28) into (26), we obtain the differential equation:

$$\frac{d\varphi}{dt} + \frac{i_1}{\tau} (\varphi_1 - \varphi) = 0, \quad (29)$$

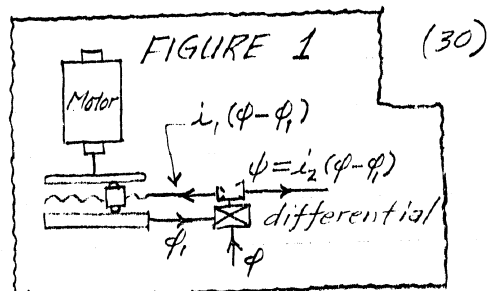
which relates the angular displacements φ_1 and φ . Comparing (29) and (15), we arrive at the conclusion that the angular displacement φ_1 will be represented by a function of time $x_1(t)$ under the following condition:

$$i_1 = k\tau.$$

It

Comparing relation (27):

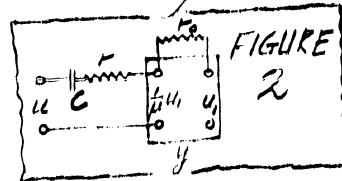
$$\frac{\psi}{\lambda} = \frac{i_2}{\lambda} (\varphi - \varphi_1)$$



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with Formula (17), we are satisfied that the ratio ψ/λ will determine in accordance with Equation (25) the derivative of angular velocity $d\phi/dt$ under the condition:

$$i_2 = k\lambda.$$



(31)

Thus, if the transmission numbers k_1 and k_2 satisfy Relations (30) and (31), the electrical feedback represented in Figure 1 determines for the input angular velocity $d\phi/dt$ its average value over the time preceding the current instant of time t (the average time is calculated with the weight $p(\tau)$ having the form in Equation (22)).

Employing in the electrical calculating instrument the capacitance C as differentiating element, we obtain in the circuit analogy of viscous friction. In the scheme represented in Figure 2, the quantity x subjected to differentiation enters the electrical differentiating unit in the form of an input electrical voltage U . The output electrical voltage U_1 determines in a given scale the derivative dU/dt :

$$U_1 = -\lambda dU/dt \quad (32)$$

where lambda λ is a given scale coefficient having the dimensions of time. If we designate by V the voltage drop in the coatings of the condenser and by r_i the input resistance of the amplifier y and, also, by μ its amplification, then we shall have, as is evident from the scheme, the following relations:

$$\frac{1}{F}(U - V - \frac{1}{\mu} U_1) + \frac{1}{F_0}(U_1 - \frac{1}{\mu} U_1) = \frac{1}{r_i} \cdot \frac{1}{\mu} U_1, \quad (33)$$

$$C \frac{dV}{dt} = \frac{1}{F}(U - V - \frac{1}{\mu} U_1).$$

Eliminating V from the Equations in (33) we arrive at the differential equation:

$$\frac{d}{dt}(-\frac{U_1}{\lambda}) + \frac{(\mu-1)r_i - r_0}{C[(\mu-1)r_i r_0 - r_0 r_i]}(-\frac{U_1}{\lambda}) = \quad (34)$$

(cont)

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$$= \frac{\mu r_0 r_2}{\lambda[(\mu-1)r_2 - r_0 - r_0 r_2]} \cdot \frac{dl_1}{dt}$$

Comparing (34) and (15) we arrive at the conclusion that the ratio $-\frac{dl_1}{\lambda}$ will, according to (32), determine the desired derivative dl_1/dt under the condition:

$$\frac{(\mu-1)r_2 - r_0}{\lambda[(\mu-1)r_2 - r_0 - r_0 r_2]} = \frac{\mu r_0 r_2}{\lambda[(\mu-1)r_2 - r_0 - r_0 r_2]} = k \quad (34')$$

where:

$$r = \frac{1}{kC} \left(1 + \frac{\lambda}{\mu}\right) \quad \text{and} \quad r_0 = \frac{\lambda}{C} \cdot \frac{1 - \frac{1}{\mu}}{1 + \frac{\lambda}{\mu kC}} \quad (35)$$

Thus, if the resistances r and r_0 satisfy the Relations in (35), the assembly represented in Figure 2 will likewise determine automatically for the desired derivative its average value over the time preceding the moment of time t (the average time is calculated with the weight $p(\tau)$ having the form in (22)). For large values of the amplification coefficient μ of the amplifier in the Formulas of (35) one can disregard the terms containing the inverse coefficient $1/\mu$. Then Formulas (35) for the calculations of the resistances r and r_0 take the simple forms:

$$r = \frac{1}{kC} \quad \text{and} \quad r_0 = \frac{\lambda}{C}.$$

¶

Paragraph 2. We shall now pass over to the problem of measuring variable velocities. In this article we shall as yet ^{assume} that the differential equation (1) is linear. Thus, let a function of time $x(t)$, whose derivatives with respect to time are subjected to measurement, satisfy the linear differential equation with constant coefficients:

$$\frac{d^n x}{dt^n} + c_n \frac{d^{n-1} x}{dt^{n-1}} + \dots + c_2 \frac{dx}{dt} + c_1 x = \text{const.} \quad (37)$$

The partial case where the differential equation (37) has the form:

has already been considered by us in the preceding section. As we have already seen, the quantity

determine the derivative $\dot{x}(t)$ with an error that decreases without limit with time if condition (35) holds.

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[illegible]
$$y_1 = a_1(x - x_1), y_2 = a_2(x - x_2), \dots, y_n = a_n(x - x_n)$$

determine respectively the derivatives dx/dt , d^2x/dt^2 , ..., $d^n x/dt^n$ with errors that decrease without limit with time, if function $x(t)$ satisfies the DE (37). Relative to the coefficients $a_1, a_2, \dots, a_n$ in the Relations (40) we shall for the present consider that they differ from zero, but are arbitrary.

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Substituting $x_1, x_2, \dots, x_n$ from (40) into (39) we obtain a system of DE: $(dy_1, a_{11}y_1 + a_{12}y_2, \dots, a_{n1}y_1 + a_{nn}y_n = dx)$

$$\left. \begin{aligned} \frac{1}{a_1} \frac{dy_1}{dt} + \frac{a_{11}}{a_1} y_1 + \frac{a_{12}}{a_2} y_2 + \dots + \frac{a_{1n}}{a_n} y_n &= \frac{dx}{dt} \\ \frac{1}{a_2} \frac{dy_2}{dt} + \frac{a_{21}}{a_1} y_1 + \frac{a_{22}}{a_2} y_2 + \dots + \frac{a_{2n}}{a_n} y_n &= \frac{dx}{dt} \\ \vdots &\vdots \\ \frac{1}{a_n} \frac{dy_n}{dt} + \frac{a_{n1}}{a_1} y_1 + \frac{a_{n2}}{a_2} y_2 + \dots + \frac{a_{nn}}{a_n} y_n &= \frac{dx}{dt} \end{aligned} \right\} \quad (41)$$

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which must be satisfied by the functions of time $y_1, y_2, \dots, y_n$.

4. Setting, furthermore, in Equation (41) the following:

$$y_1 = \frac{dx}{dt} + \delta_1, \quad y_2 = \frac{d^2x}{dt^2} + \delta_2, \quad \dots, \quad y_n = \frac{d^nx}{dt^n} + \delta_n, \quad (42)$$

we obtain the following system of DE:

$$\left. \begin{aligned} \frac{1}{a_1} \frac{d\delta_1}{dt} + \frac{a_{11}}{a_1} \delta_1 + \frac{a_{12}}{a_1} \delta_2 + \dots + \frac{a_{1n}}{a_1} \delta_n &= \\ &= \left(1 - \frac{a_{11}}{a_1}\right) \frac{dx}{dt} - \left(\frac{1}{a_1} - \frac{a_{12}}{a_1}\right) \frac{d^2x}{dt^2} - \frac{a_{13}}{a_1} \frac{d^3x}{dt^3} - \dots - \frac{a_{1n}}{a_1} \frac{d^nx}{dt^n}, \\ \frac{1}{a_2} \frac{d\delta_2}{dt} + \frac{a_{21}}{a_2} \delta_1 + \frac{a_{22}}{a_2} \delta_2 + \dots + \frac{a_{2n}}{a_2} \delta_n &= \\ &= \left(1 - \frac{a_{21}}{a_1}\right) \frac{dx}{dt} - \frac{a_{22}}{a_2} \frac{d^2x}{dt^2} - \left(\frac{1}{a_2} - \frac{a_{23}}{a_2}\right) \frac{d^3x}{dt^3} - \dots - \frac{a_{2n}}{a_n} \frac{d^nx}{dt^n}, \\ \dots \dots \dots \\ \frac{1}{a_n} \frac{d\delta_n}{dt} + \frac{a_{n1}}{a_n} \delta_1 + \frac{a_{n2}}{a_n} \delta_2 + \dots + \frac{a_{nn}}{a_n} \delta_n &= \\ &= \left(1 - \frac{a_{n1}}{a_1}\right) \frac{dx}{dt} - \frac{a_{n2}}{a_2} \frac{d^2x}{dt^2} - \dots - \frac{a_{nn}}{a_n} \frac{d^nx}{dt^n} - \frac{1}{a_n} \frac{d^{n+1}x}{dt^{n+1}}. \end{aligned} \right\} \quad (43)$$

As is evident from (43), if one sets:

$$\left. \begin{aligned} a_{11} &= a_1, \quad a_{12} = -\frac{a_2}{a_1}, \quad a_{13} = 0, \quad \dots, \quad a_{1n} = 0 \\ a_{21} &= a_1, \quad a_{22} = 0, \quad a_{23} = -\frac{a_3}{a_2}, \quad \dots, \quad a_{2n} = 0 \\ \dots \dots \dots \\ a_{n1} &= a_1 \left(1 + \frac{a_1}{a_n}\right), \quad a_{n2} = a_2 \frac{a_1}{a_n}, \quad \dots, \quad a_{n, n-1} = a_{n-1} \frac{a_{n-1}}{a_n}, \quad a_{nn} = a_n \end{aligned} \right\} \quad (44)$$

then the system of DE in (43) becomes homogeneous for condition (37), and will have the partial solution:

$$\delta_1 = 0, \quad \delta_2 = 0, \quad \dots, \quad \delta_n = 0 \quad (44')$$

The system of DE in (41) will in this case have the partial solution:

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$$y_1 = \frac{dx}{dt}, y_2 = \frac{d^2x}{dt^2}, \dots, y_n = \frac{d^n x}{dt^n} \quad (44'')$$

Substituting (44) into (37), (41), (43), we obtain for the functions of time $x, x_1, \dots, x_n, y_1, y_2, \dots, y_n, \delta_1, \delta_2, \dots, \delta_n$ the DE:

$$\left. \begin{aligned} \frac{dx_1}{dt} + a_1(x_1 - x) - \frac{c_1}{a_1}(x_2 - x) &= 0 \\ \frac{dx_2}{dt} + a_1(x_1 - x) - \frac{c_2}{a_2}(x_2 - x) &= 0 \\ &\dots\dots\dots \\ \frac{dx_{n-1}}{dt} + a_{n-1}(x_{n-1} - x) - \frac{c_{n-1}}{a_{n-1}}(x_n - x) &= 0 \\ \frac{dx_n}{dt} + a_n(1 + \frac{c_1}{a_n})(x_1 - x) + a_n \frac{c_2}{a_n}(x_2 - x) + \dots + c_n(x_n - x) &= 0 \end{aligned} \right\} \quad (45)$$

$$\left. \begin{aligned} \frac{dy_1}{dt} + a_1 y_1 - y_2 &= c_1 \frac{dx}{dt} \\ \frac{dy_2}{dt} + a_2 y_2 - y_3 &= c_2 \frac{dx}{dt} \\ &\dots\dots\dots \\ \frac{dy_{n-1}}{dt} + a_{n-1} y_{n-1} - y_n &= c_{n-1} \frac{dx}{dt} \\ \frac{dy_n}{dt} + (a_n + c_1) y_1 + c_2 y_2 + \dots + c_n y_n &= c_n \frac{dx}{dt} \end{aligned} \right\} \quad (46)$$

$$\left. \begin{aligned} \frac{d\delta_1}{dt} + a_1 \delta_1 - \delta_2 &= 0 \\ \frac{d\delta_2}{dt} + a_2 \delta_2 - \delta_3 &= 0 \\ &\dots\dots\dots \\ \frac{d\delta_{n-1}}{dt} + a_{n-1} \delta_{n-1} - \delta_n &= 0 \\ \frac{d\delta_n}{dt} + (a_n + c_1) \delta_1 + c_2 \delta_2 + \dots + c_n \delta_n &= - \left(c_1 \frac{dx}{dt} + c_2 \frac{d^2x}{dt^2} + \dots + c_n \frac{d^n x}{dt^n} + \frac{d^{n+1}x}{dt^{n+1}} \right) \end{aligned} \right\} \quad (47)$$

As we have already indicated above, if the equation $x(t)$ satisfies the DE (37), then the system of DE (47) becomes homogeneous. We shall find for this case the general solution of this system of DE. The characteristic equation of this system can be represented in the form:

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$$\begin{vmatrix} a_n + c_1 & c_1 & c_3 & c_4 & \dots & c_{n-2} & c_{n-1} & c_n + k \\ -(a_1 + k) & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -a_2 & -k & 1 & 0 & \dots & 0 & 0 & 0 \\ -a_3 & 0 & -k & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{n-2} & 0 & 0 & 0 & \dots & -k & 1 & 0 \\ -a_{n-1} & 0 & 0 & 0 & \dots & 0 & -k & 1 \end{vmatrix} = 0. \quad (48)$$

Expanding the determinant we find for Equation (48) the following form:

$$\begin{aligned} & a_n + c_1 + c_1(a_1 + k) + c_3[(a_1 + k)k + a_2] + c_4[(a_1 + k)k^2 + a_2k + a_3] + \dots + \\ & c_{n-1}[(a_1 + k)k^{n-3} + a_2k^{n-4} + a_3k^{n-5} + \dots + a_{n-3}k + a_{n-2}] + \\ & + (c_n + k)[(a_1 + k)k^{n-2} + a_2k^{n-3} + a_3k^{n-4} + \dots + a_{n-2}k + a_{n-1}] = 0 \end{aligned}$$

or

$$\begin{aligned} & k^n + (a_1 + c_n)k^{n-1} + (a_2 + a_1c_n + c_{n-1})k^{n-2} + (a_3 + a_2c_n + a_1c_{n-1} + c_{n-2})k^{n-3} + \dots \\ & \dots + (a_{n-1} + a_{n-2}c_n + a_{n-3}c_{n-1} + \dots + a_1c_3 + c_2)k + \\ & a_1 + a_{n-1}c_n + a_{n-2}c_{n-1} + \dots + a_1c_2 + c_1 = 0. \end{aligned} \quad (49)$$

The coefficients $a_1, a_2, \dots, a_n$, which up to now we have assumed to be arbitrary, requiring only that they be different from zero, are connected with the roots of Equation (49) $k_1, k_2, \dots, k_n$ by the relations:

$$\left. \begin{aligned} a_1 &= -(c_n + \sum_{i=1}^n k_i) \\ a_2 &= -(c_{n-1} + a_1c_n - \sum_{i=1}^n \sum_{j=i+1}^n k_i k_j) \\ \dots & \dots \\ a_n &= -[c_1 + a_1c_2 + \dots + a_{n-2}c_{n-1} + a_{n-1}c_n (-1)^{n-1} \prod_{i=1}^n k_i] \end{aligned} \right\} (50)$$

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Given the roots of the characteristic equation $k_1, k_2, k_3, \dots, k_n$, we can consequently calculate from the relations in (50) those values of the coefficients $a_1, a_2, \dots, a_n$ for which the characteristic equation possesses the given roots. Later we shall consider the roots of the characteristic equation real, negative and different. In order that the coefficients $a_1, a_2, \dots, a_n$ be different from zero it is necessary that the following inequalities hold:

$$\sum_{i=1}^n k_i + c_n \neq 0, \quad \sum_{i=1}^n k_i \sum_{j=1}^n (k_j + c_n) + c_n^2 - c_{n-1} \neq \text{etc.}$$

they are for the roots of the characteristic equation k_i can be arbitrary.

By way of successive calculations we find that the general solution of the differential equations in (47) has the following form:

$$\left. \begin{aligned} S_1 &= \sum_{i=1}^n C_i e^{k_i t} \\ S_2 &= \sum_{i=1}^n (a_1 + k_i) C_i e^{k_i t} \\ S_3 &= \sum_{i=1}^n (a_2 + a_1 k_i + k_i^2) C_i e^{k_i t} \\ &\dots \dots \dots \\ S_n &= \sum_{i=1}^n (a_{n-1} + a_{n-2} k_i + \dots + a_1 k_i^{n-2} + k_i^{n-1}) C_i e^{k_i t} \end{aligned} \right\} (52)$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants if the function $x(t)$ satisfies the DE (37). Substitution (52) in (42) we find that the general solution of the system of DE's in (46) will in this case have the following form:

$$\left. \begin{aligned} y_1 &= \frac{dx}{dt} + \sum_{i=1}^n C_i e^{k_i t} \\ y_2 &= \frac{d^2x}{dt^2} + \sum_{i=1}^n (a_1 + k_i) C_i e^{k_i t} \\ &\dots \dots \dots \\ y_n &= \frac{d^nx}{dt^n} + \sum_{i=1}^n (a_{n-1} + a_{n-2} k_i + \dots + a_1 k_i^{n-2} + k_i^{n-1}) C_i e^{k_i t} \end{aligned} \right\} (53)$$

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Thus, if we are given n different negative numbers $k_1, k_2, \dots, k_n$ satisfying the inequalities (51) and calculate further the coefficients $a_1, a_2, \dots, a_n$ according to formula (50) and if we adjust the integrator that works out the solution of the DE's in (45) and determines the functions of time $y_1, y_2, \dots, y_n$ in agreement with the relations in (40), then these time functions will correspondingly determine the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}$ with errors that decrease without limit with time, if the time functions $x(t)$ satisfy the DE's in (37). Actually, the relations in (53) indicate that in this case the errors of measurement of the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}$ tend toward zero when time t increases without limit.

Now we shall proceed to an analysis of the errors which arise in the determination of the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}$ in the case where the time functions $x(t)$ enter the integrator with a certain error $\delta x(t)$. From the first $n-1$ equations (46) we find successively:

$$\left. \begin{aligned} y_2 &= a_1 y_1 + \frac{dy_1}{dt} - a_1 \frac{dx}{dt} \\ y_3 &= a_2 y_1 + a_1 \frac{dy_1}{dt} + \frac{d^2 y_1}{dt^2} - a_2 \frac{dx}{dt} - a_1 \frac{d^2 x}{dt^2} \\ &\dots\dots\dots \\ y_n &= a_{n-1} y_1 + a_{n-2} \frac{d^2 y_1}{dt^2} + \dots + a_1 \frac{d^{n-2} y_1}{dt^{n-2}} + \frac{d^{n-1} y_1}{dt^{n-1}} \\ &\quad - a_{n-1} \frac{dx}{dt} - a_{n-2} \frac{d^2 x}{dt^2} - \dots - a_1 \frac{d^{n-1} x}{dt^{n-1}} \end{aligned} \right\} \quad (54)$$

Substituting (54) into last of equations (46), we obtain the following DE, (55) and (56) respectively:

$$\left\{ \begin{aligned} &\frac{d^ny_1}{dt^n} + (a_1 + c_n) \frac{d^{n-1}y_1}{dt^{n-1}} + (a_2 + a_1 c_n + c_{n-1}) \frac{d^{n-2}y_1}{dt^{n-2}} + \dots + (a_n + a_{n-1} c_n + \dots + a_1 c_2 + c_1) y_1 = X(t) \\ &X(t) = a_1 \frac{d^nx}{dt^n} + (a_2 + a_1 c_n) \frac{d^{n-1}x}{dt^{n-1}} + (a_3 + a_2 c_n + a_1 c_{n-1}) \frac{d^{n-2}x}{dt^{n-2}} + \dots + \\ &\quad + (a_n + a_{n-1} c_n + a_{n-2} c_{n-1} + \dots + a_1 c_2) \frac{dx}{dt} \end{aligned} \right. \quad (55) \quad (56)$$

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The characteristic equation (49) in DE (55) has the roots $k_1, k_2, \dots, k_n$; if we follow the method of variation of parameters and seek the solution of equation (55) in the form:

$$y_i = \sum_{k=1}^n f_k(t) \cdot e^{k_i t} \quad (57)$$

then for functions $f_1(t), f_2(t), \dots, f_n(t)$ we obtain the following equations:

$$\left. \begin{aligned} \sum_{k=1}^n f'_k(t) \cdot k_i^{n-1} e^{k_i t} &= X(t) \\ \sum_{k=1}^n f'_k(t) k_i^{n-2} e^{k_i t} &= 0 \\ &\dots\dots\dots \\ \sum_{k=1}^n f'_k(t) k_i e^{k_i t} &= 0 \end{aligned} \right\} \quad (58)$$

From equations (58) we find (59) following:

$$f'_k(t) e^{k_i t} = \begin{vmatrix} k_1^{n-1} & k_2^{n-1} & \dots & k_{i-1}^{n-1} & X(t) & k_{i+1}^{n-1} & \dots & k_n^{n-1} \\ k_1^{n-2} & k_2^{n-2} & \dots & k_{i-1}^{n-2} & 0 & k_{i+1}^{n-2} & \dots & k_n^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \end{vmatrix}$$

$$\begin{vmatrix} k_1^{n-1} & k_2^{n-1} & \dots & k_n^{n-1} \\ k_1^{n-2} & k_2^{n-2} & \dots & k_n^{n-2} \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{vmatrix}$$

$$= (-1)^{i+1} \frac{K_i}{K} X(t) \quad (59)$$

where K_i and K are the upper and lower determinants respectively. (60)

$$\text{From (59) we find:} \\ f_k(t) = (-1)^{i+1} \frac{K_i}{K} \int_{-\infty}^t X(s) e^{-k_i s} ds + C_i. \quad (61)$$

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Substituting (61) in (57) we obtain

$$y_i(t) = \sum_{i=1}^n (-1)^{i+1} \frac{K_i}{K} \int_{-\infty}^t X(s) e^{k_i(t-s)} ds + \sum_{i=1}^n C_i e^{k_i t}$$

or setting $s = t - \tau$:

$$y_i(t) = \sum_{i=1}^n (-1)^{i+1} \frac{K_i}{K} \int_0^{\infty} X(t-\tau) e^{k_i \tau} d\tau + \sum_{i=1}^n C_i e^{k_i t} \quad (62)$$

For $j = 2, 3, \dots, (n-1), n$ we find (63) following:

$$(A) \sum_{i=1}^n (-1)^{i+1} K_i \int_0^{\infty} \frac{d^j x}{dt^j} (t-\tau) e^{k_i \tau} d\tau = \sum_{i=1}^n (-1)^{i+1} K_i [Q]$$

$$\left(\text{where } Q \equiv \frac{d^j x}{dt^j} (t) + k_i \int_0^{\infty} \frac{d^{j-1} x}{dt^{j-1}} (t-\tau) e^{k_i \tau} d\tau \right);$$

$$\begin{aligned} \text{A similar} \quad & \sum_{i=1}^n (-1)^{i+1} K_i \left[\frac{d^{j-1} x}{dt^{j-1}} (t) + k_i \frac{d^{j-2} x}{dt^{j-2}} (t) + k_i^2 \int_0^{\infty} \frac{d^{j-2} x}{dt^{j-2}} (t-\tau) e^{k_i \tau} d\tau \right] \\ &= \sum_{i=1}^n (-1)^{i+1} K_i \left[\frac{d^{j-1} x}{dt^{j-1}} (t) + k_i \frac{d^{j-2} x}{dt^{j-2}} (t) + \dots + k_i^{j-2} \frac{dx}{dt} (t) + k_i^{j-1} \int_0^{\infty} \frac{dx}{dt} (t-\tau) e^{k_i \tau} d\tau \right] \\ &= \sum_{i=1}^n (-1)^{i+1} K_i k_i^{j-1} \int_0^{\infty} \frac{dx}{dt} (t-\tau) e^{k_i \tau} d\tau, \quad (63) \end{aligned}$$

Since

$$\sum_{i=1}^n (-1)^{i+1} K_i k_i^m = \begin{vmatrix} k_1^m & k_2^m & \dots & k_n^m \\ k_1^{m-1} & k_2^{m-1} & \dots & k_n^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{m-n+1} & k_2^{m-n+1} & \dots & k_n^{m-n+1} \end{vmatrix} = 0 \quad \text{for } m = 0, 1, \dots, (n-2)$$

Substituting (56) into (62) and keeping (63) in mind, we find (64) following:

$$\begin{aligned} y_i(t) &= \sum_{i=1}^n (-1)^{i+1} \frac{K_i}{K} \int_0^{\infty} \frac{dx}{dt} (t-\tau) [a_1 k_i^{n-1} + (a_2 + a_1 c_n) k_i^{n-2} \\ &\quad + (a_3 + a_2 c_n + a_1 c_{n-1}) k_i^{n-3} + \dots + a_n + a_{n-1} c_n + a_{n-2} c_{n-1} + \dots + a_1 c_2] e^{k_i \tau} d\tau \\ &\quad + \sum_{i=1}^n C_i e^{k_i t} = \sum_{i=1}^n (-1)^i \frac{K_i}{K} \int_0^{\infty} \frac{dx}{dt} (t-\tau) (k_i^n + c_n k_i^{n-1} + c_{n-1} k_i^{n-2} + \dots \\ &\quad + \dots + c_1) e^{k_i \tau} d\tau + \sum_{i=1}^n C_i e^{k_i t}, \quad (64) \end{aligned}$$

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since $k_1, k_2, \dots, k_n$ are the roots of equation (49). Thus, for a steady state, that is, for a reduction almost to zero of the sum

$$\sum_{i=1}^n C_i e^{k_i t},$$

the derivative $\frac{dx}{dt}$ will be determined by the following:

$$y_1(t) = \int_0^\infty \frac{dx}{dt}(t-\tau) p_1(\tau) d\tau \quad (65)$$

$$\text{where } p_1(\tau) = \sum_{i=1}^n (-1)^i \frac{K_i}{K} (k_i^n + c_n k_i^{n-1} + c_{n-1} k_i^{n-2} + \dots + c_2 k_i + c_1) e^{k_i \tau} \quad (66)$$

That is, the average value of the derivative $\frac{dx}{dt}$ will be determined over the time preceding the current moment of time t and will be calculated with the weight $p_1(\tau)$ which has the form (66). If the function $x(t)$ enters ("inputs") the integrator with an error $\delta x(t)$, then in the determination of the derivative $\frac{dx}{dt}$ we obtain for the steady state the error:

$$\Delta_1(t) = \int_0^\infty \frac{d\delta x}{dt}(t-\tau) p_1(\tau) d\tau = \delta x(t) p_1(0) + \int_0^\infty \delta x(t-\tau) dp_1(\tau) \quad (67)$$

$$\text{If } |\delta x(\tau)| \leq \epsilon \text{ for } \tau \leq t$$

$$\text{then } |\Delta_1(t)| \leq \epsilon [|p_1(0)| + \int_0^\infty |dp_1(\tau)|]$$

$$\text{or } |\Delta_1(t)| \leq \epsilon \phi_1(k_1, k_2, \dots, k_n) \quad (68)$$

$$\text{where } \phi_1(k_1, k_2, \dots, k_n) = |p_1(0)| + \int_0^\infty |dp_1(\tau)| \quad (69)$$

In a similar way one can evaluate the error with which in this case the higher derivatives $\frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}$ are determined. However, utilizing the formulas obtained above for the first derivative we can obtain the necessary evaluation by simpler means, which we shall employ below.

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A homogeneous system of DE's corresponding to equations (47) possesses the general solution (52). Since the homogeneous systems of DE's corresponding (systems) to equations (46) for functions $y_1, y_2, \dots, y_n$ and equations (47) for functions $s_1, s_2, \dots, s_n$ are identical, we can immediately find, by the method of variation of parameters, the solution of equations (46) in the following form:

$$\left. \begin{aligned} y_1 &= \sum_{i=1}^n \psi_i(t) e^{k_i t} \\ y_2 &= \sum_{i=1}^n (a_1 + k_i) \psi_i(t) e^{k_i t} \\ y_3 &= \sum_{i=1}^n (a_2 + a_1 k_i + k_i^2) \psi_i(t) e^{k_i t} \\ &\dots \dots \dots \\ y_n &= \sum_{i=1}^n (a_{n-1} + a_{n-2} k_i + a_{n-3} k_i^2 + \dots + a_1 k_i^{n-2} + k_i^{n-1}) \psi_i(t) e^{k_i t} \end{aligned} \right\} (70)$$

The functions $\psi_1(t), \psi_2(t), \dots, \psi_n(t)$ are obtained for this equation in the following form:

$$\left. \begin{aligned} \sum_{i=1}^n \psi_i'(t) e^{k_i t} &= a_1 \frac{dx}{dt} \\ \sum_{i=1}^n (a_1 + k_i) \psi_i'(t) e^{k_i t} &= a_2 \frac{dx}{dt} \\ &\dots \dots \dots \\ \sum_{i=1}^n (a_{n-1} + a_{n-2} k_i + a_{n-3} k_i^2 + \dots + a_1 k_i^{n-2} + k_i^{n-1}) \psi_i'(t) e^{k_i t} &= a_n \frac{dx}{dt} \end{aligned} \right\} (71)$$

The solution of the above equations (71) has the form:

$$\psi_i'(t) = b_i \frac{dx}{dt} e^{-k_i t} \quad (i = 1, 2, \dots, n) \quad (72)$$

Carrying out the integration according to (72), we find:

$$\psi_i(t) = b_i \int_{-\infty}^t \frac{dx}{dt}(s) e^{-k_i s} ds + C_i \quad (i = 1, 2, \dots, n), \quad (73)$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants. Substituting (73) into (70), we find:

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$$\left. \begin{aligned}
 (A): y_j(t) &= \sum_{i=1}^n (a_{j-1} + a_{j-2}k_i + a_{j-3}k_i^2 + \dots + a_1k_i^{j-2} + k_i^{j-1}) \cdot [Q_i] \\
 (\text{where } Q_i &\equiv b_i \int_{-\infty}^t \frac{dx}{dt}(s) e^{k_i(t-s)} ds + C_i e^{k_i t}) \\
 A \text{ is equal to } &\sum_{i=1}^n (a_{j-1} + a_{j-2}k_i + a_{j-3}k_i^2 + \dots + a_1k_i^{j-2} + k_i^{j-1}) [R_i] \\
 (\text{where } R_i &\equiv b_i \int_0^{\infty} \frac{dx}{dt}(t-\tau) e^{k_i \tau} d\tau + C_i e^{k_i t}), j = 1, 2, \dots, n.
 \end{aligned} \right\} (74)$$

Thus, for the derivative $\frac{dx}{dt}$ in a steady state, we obtain the following expression:

$$y_j(t) = \int_{-\infty}^t \frac{dx}{dt}(t-\tau) p_j(\tau) d\tau \quad (75)$$

$$\text{where } p_j(\tau) = \sum_{i=1}^n (a_{j-1} + a_{j-2}k_i + a_{j-3}k_i^2 + \dots + a_1k_i^{j-2} + k_i^{j-1}) b_i e^{k_i \tau} \quad (76).$$

That is, we obtain the average value of the derivative $\frac{dx}{dt}$ over the time preceding the current moment of time t and with the weight function $p_j(\tau)$ given by formula (76). Since

$$p(\tau) = \sum_{i=1}^n b_i e^{k_i \tau},$$

then it follows from relation (76) that:

$$p_j(\tau) = \sum_{i=1}^n [a_{j-1} p_i(\tau) + a_{j-2} p_i'(\tau) + \dots + a_1 p_i^{(j-2)}(\tau) + p_i^{(j-1)}(\tau)] \quad (77)$$

where $j = 2, 3, \dots, n$.

But the weight function $p_i(\tau)$ was already found by us above. Substituting (46) into (77) we find:

$$\left. \begin{aligned}
 p_j(\tau) &= \sum_{i=1}^n (-1)^i \frac{K_i}{K} (k_i^n + c_n k_i^{n-1} + c_{n-1} k_i^{n-2} + \dots + c_2 k_i + c_1) \times \\
 &\quad \times (a_{j-1} + a_{j-2}k_i + a_{j-3}k_i^2 + \dots + a_1k_i^{j-2} + k_i^{j-1}) e^{k_i \tau}
 \end{aligned} \right\} (78)$$

where $j = 2, 3, \dots, n$.

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Thus, if the function $x(t)$ enters ("inputs") the integrator with the error $\delta x(t)$, then the errors $\Delta_1(t), \Delta_2(t), \dots, \Delta_n(t)$ in the determination of the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}$ will satisfy the inequalities:

$$|\Delta_j(t)| \leq \varepsilon \Phi_j(k_1, k_2, \dots, k_n), \quad j=1, 2, \dots, n, \quad (79)$$

where $\varepsilon = \max |\delta x(t)|$, and

$$\Phi_j(k_1, k_2, \dots, k_n) = |p_j(0)| + \int_0^\infty |dp_j(\tau)|, \quad j=1, 2, \dots, n, \quad (80)$$

and functions $p_1(\tau), p_2(\tau), \dots, p_n(\tau)$ are determined by relations (66) and (78). Since the functions $\Phi_1, \Phi_2, \dots, \Phi_n$ for the given output DE (37) depends only upon the quantities $k_1, k_2, \dots, k_n$, then fixing the values of these quantities we can guarantee completely a definite accuracy of determination of the function $x(t)$ no matter what kind of character the error $\delta x(t)$ possesses.

For the partial case where

$$c_1 = c_2 = c_3 = \dots = c_n = 0; \quad (81)$$

that is, where the output DE (37) has the form:

$$\frac{d^nx}{dt^n} = \text{constant}, \quad (82)$$

the functions $\Phi_j(k_1, k_2, \dots, k_n)$ are homogeneous functions in the arguments $k_1, k_2, \dots, k_n$ of degree j ($j=1, 2, \dots, n$). Actually, if the quantities $k_1, k_2, \dots, k_n$ are replaced by the quantities $\delta k_1, \delta k_2, \dots, \delta k_n$, then as is obvious from (50) and (60) the functions of the arguments $k_1, k_2, \dots, k_n$

namely a_i and $\frac{K_i}{K}$ ($i=1, 2, \dots, n$)

pass over to:

$$\delta^i a_i \quad \text{and} \quad \delta^{1-n} \frac{K_i}{K} \quad (i=1, 2, \dots, n) \quad [*sic!].$$

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And the functions $\varphi_j(\tau)$ ($j=1,2,\dots,n$) pass over to the functions:

$$\partial_j \varphi_j(\partial \tau) \quad (j=1,2,\dots,n),$$

as relations (66) and (78) indicate. From (80) we find:

$$\varphi_j(\partial k_1, \partial k_2, \dots, \partial k_n) = \partial \partial \varphi_j(k_1, k_2, \dots, k_n) \quad (j=1,2,\dots,n) \quad (83)$$

which was to be demonstrated.

In this manner, ^{by} taking $k_1, k_2, \dots, k_n$ sufficiently small in absolute value we can on the basis of (79) guarantee any given accuracy of determination of the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}$ ^{namely} what the specified accuracy of determination of the function $x(t)$ may be. However, as is seen from (53), the less the absolute value of the quantities $k_1, k_2, \dots, k_n$ the more slowly the functions $e^{k_1 t}, e^{k_2 t}, \dots, e^{k_n t}$ decrease with time and the more prolonged the unstable state.

In the general case, to a given "output" DE (37) there will correspond positive constants (etc) $\eta_1, \eta_2, \dots, \eta_n$ bounding below the functions $\varphi_1, \varphi_2, \dots, \varphi_n$.

$$\varphi_j(k_1, k_2, \dots, k_n) \geq \eta_j > 0 \quad (j=1,2,\dots,n) \quad (84)$$

and to a given accuracy of determination of the function $x(t)$, namely ε , there will correspond specific attainable accuracies of determination of the derivatives

$$\frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}, \text{ namely } \eta_1 \varepsilon, \eta_2 \varepsilon, \dots, \eta_n \varepsilon.$$

Paragraph 3: We shall now illustrate, by an example, the general considerations set forth by us in the preceding Paragraph.

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Let the "output" DE (37) have the form:

$$\frac{d^2x}{dt^2} = \text{constant} \quad (85)$$

and, for this condition, it is required that the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ be determined. In the given example, $n=2$ and $c_1=c_2=0$ and on the basis of (40) and (45) we can state that the functions of time

$$y_1 = a_1(x - x_1) \text{ and } y_2 = a_2(x - x_2) \quad (86)$$

will, for the steady state, determine the desired derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, if x_1 and x_2 satisfy the following system of DE's:

$$\left. \begin{aligned} \frac{dx_1}{dt} + a_1(x_1 - x) - \frac{a_2}{a_1}(x_2 - x) &= 0 \\ \frac{dx_2}{dt} + a_1(x_1 - x) &= 0 \end{aligned} \right\} \quad (87)$$

The system of DE's (46) for the time functions y_1 and y_2 in the given case will have the form:

$$\left. \begin{aligned} \frac{dy_1}{dt} + a_1 y_1 - y_2 &= a_1 \frac{dx}{dt} \\ \frac{dy_2}{dt} + a_2 y_2 &= a_2 \frac{dx}{dt} \end{aligned} \right\} \quad (88)$$

We shall examine the question of the accuracy with which the desired derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ will be determined, in this case, if the time function $x(t)$ is inserted into the differentiating unit (input terminal) of the apparatus with an error $\delta x(t)$. If we designate by $\Delta_1(t)$ and $\Delta_2(t)$ the errors that arise for this error in the determination of the sought-for derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, then according to (79) the following inequalities will hold:

$$|\Delta_1(t)| \leq \varepsilon \varphi_1(k_1, k_2) \text{ and } |\Delta_2(t)| \leq \varepsilon \varphi_2(k_1, k_2) \quad (89)$$

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where $\varepsilon = \max |\delta x(t)|$, and the functions $\phi_1(k_1, k_2)$ and $\phi_2(k_1, k_2)$ according to (80), (66), (78) and (60) are determined by the relations:

$$\left. \begin{aligned} \phi_1(k_1, k_2) &= |p_1(0)| + \int_0^\infty |p_1'(\tau)| d\tau, \\ \phi_2(k_1, k_2) &= |p_2(0)| + \int_0^\infty |p_2'(\tau)| d\tau, \end{aligned} \right\} \quad (90)$$

$$\left. \begin{aligned} \text{where } p_1(\tau) &= \frac{1}{k_1 - k_2} (-k_1^2 e^{k_1 \tau} + k_2^2 e^{k_2 \tau}) \\ p_2(\tau) &= \frac{1}{k_1 - k_2} [-k_1^2 (a_1 + k_1) e^{k_1 \tau} + k_2^2 (a_1 + k_2) e^{k_2 \tau}] \end{aligned} \right\} \quad (91)$$

Formulas (50) in our example establishes the following connection between the coefficients a_1, a_2 and k_1, k_2 :

$$a_1 = -(k_1 + k_2) \quad \text{and} \quad a_2 = k_1 k_2. \quad (92)$$

Since, according to the condition, $k_1 < 0$ and $k_2 < 0$, then we have:

$$\left. \begin{aligned} p_1(0) > 0, \quad p_2(0) > 0, \quad p_1'(\tau) < 0 \quad \text{for } \tau < \tau_1; \\ p_2'(\tau) < 0 \quad \text{for } \tau < \tau_2; \quad p_2'(\tau) > 0 \quad \text{for } \tau > \tau_2, \end{aligned} \right\} \quad (93)$$

where τ_1 and τ_2 are the solutions of the equations $p_1'(\tau) = 0$ and $p_2'(\tau) = 0$; that is,

$$\tau_1 = \frac{3}{k_1 - k_2} \ln \frac{k_2}{k_1} \quad \text{and} \quad \tau_2 = \frac{2}{k_1 - k_2} \ln \frac{k_2}{k_1}. \quad (94)$$

According to (93) we can give to formulas (90) the form:

$$\phi_1(k_1, k_2) = p_1(0) - \int_0^{\tau_1} p_1'(\tau) d\tau + \int_{\tau_1}^\infty p_1'(\tau) d\tau = 2[p_1(0) - p_1(\tau_1)]$$

$$\text{and similarly} \quad \phi_2(k_1, k_2) = 2[p_2(0) - p_2(\tau_2)]$$

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or on the basis of (91) and (94):

$$\begin{aligned}\phi_1(k_1, k_2) &= 2 \left\{ -(k_1 + k_2) + \frac{1}{k_1 - k_2} \left[k_1^2 \left(\frac{k_2}{k_1} \right)^{\frac{3k_1}{k_1 - k_2}} - k_2^2 \left(\frac{k_1}{k_2} \right)^{\frac{3k_2}{k_1 - k_2}} \right] \right\} \\ \phi_2(k_1, k_2) &= 2k_1 k_2 \left\{ 1 - \frac{1}{k_1 - k_2} \left[k_1 \left(\frac{k_2}{k_1} \right)^{\frac{2k_1}{k_1 - k_2}} - k_2 \left(\frac{k_1}{k_2} \right)^{\frac{2k_2}{k_1 - k_2}} \right] \right\} \quad (95)\end{aligned}$$

where ε_1 and ε_2 are the given accuracies of determination of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$; that is, if it is required to satisfy the conditions:

$$|\Delta_1(t)| \leq \varepsilon_1 \quad \text{and} \quad |\Delta_2(t)| \leq \varepsilon_2,$$

then according to (89) the functions $\phi_1(k_1, k_2)$ and $\phi_2(k_1, k_2)$ must satisfy the inequalities

$$\phi_1(k_1, k_2) \leq \frac{\varepsilon_1}{\varepsilon} \quad \text{and} \quad \phi_2(k_1, k_2) \leq \frac{\varepsilon_2}{\varepsilon} \quad (96)$$

As an invariant from relations (53), in the general case the errors of determination of the derivatives $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, ... $\frac{d^nx}{dt^n}$ for an unsteady state are linear combinations of the expressions $e^{k_1 t}$, $e^{k_2 t}$, ..., $e^{k_n t}$. Thus, an unsteady state will proceed more rapidly the greater becomes the least of the absolute values of the coefficients $k_1, k_2, \dots, k_n$. Let us agree in our example that $|k_1| < |k_2|$ and assume:

$$k_1 = -k \quad \text{and} \quad k_2 = -\frac{k}{\vartheta} \quad (k > 0, 0 < \vartheta < 1) \quad (97)$$

Let us now select the coefficients such that they satisfy the inequalities (96) for the maximum value k ; that is, so that they guarantee the required accuracy of determination of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ for the most advantageous character of the unsteady state.

Substituting (97) in (95), we find:

$$\phi_1 = k f_1(\vartheta) \quad \text{and} \quad \phi_2 = k^2 f_2(\vartheta) \quad (98)$$

$$\text{where} \quad f_1(\vartheta) = 2 \left(1 + \frac{1}{\vartheta} + \vartheta^{\frac{1+2\vartheta}{1-\vartheta}} \right) \quad \text{and} \quad f_2(\vartheta) = 2 \left(\frac{1}{\vartheta} + \vartheta^{\frac{2\vartheta}{1-\vartheta}} \right) \quad (99)$$

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In the interval $(0,1)$ both functions $f_1(\theta)$ and $f_2(\theta)$ acquire minimum values for $\theta=1$. Thus, to given values of the functions ϕ_1 and ϕ_2 then correspond maximum values of k for $\theta=1$. Setting in (98) $\theta=1$, we find

$$\phi_1 = k f_1(1) = 2k(2 + \frac{1}{e^3}) \text{ and } \phi_2 = k^2 f_2(1) = 2k^2(1 + \frac{1}{e^2}) \quad (100)$$

Substituting (100) into (96) we obtain the following inequalities which must be satisfied by the coefficient k :

$$k \leq \frac{\epsilon_1}{4\epsilon(1 + \frac{1}{2e^3})} \text{ and } k \leq \sqrt{\frac{\epsilon_2}{2\epsilon(1 + \frac{1}{e^2})}} \quad (101)$$

$$\text{Thus setting } k_1 = k_2 = k \quad (102)$$

$$\text{where } k = \min \left[\frac{\epsilon_1}{4\epsilon(1 + \frac{1}{2e^3})}, \sqrt{\frac{\epsilon_2}{2\epsilon(1 + \frac{1}{e^2})}} \right] \quad (103)$$

we can guarantee the required accuracy of determination of the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ by ensuring at the same time the most advantageous character of the unsteady state. Substituting (102) into (92) we find that the coefficients a_1 and a_2 will in this case equal

$$a_1 = 2k \text{ and } a_2 = k^2 \quad (104)$$

The weight functions $p_1(\tau)$ and $p_2(\tau)$ will in this case, in accordance with (91), have the form:

$$p_1(\tau) = k(2 - k\tau) e^{-k\tau} \text{ and } p_2(\tau) = k^2(1 - k\tau) e^{-k\tau} \quad (105)$$

We are easily convinced by direct calculations that for an unsteady state and for condition (85), that is for $x = a + b t + t^2$, we shall have in accordance with (75) and (105) the following values for y_1 and y_2 :

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$$y_1 = \int_0^{\infty} [b+2c(t-\tau)] p_1(\tau) d\tau = \int_0^{\infty} [b+2c(t-\tau)] k(2-k\tau) e^{-k\tau} d\tau$$

$$\underline{= b+2ct}$$

$$y_2 = \int_0^{\infty} [b+2c(t-\tau)] p_2(\tau) d\tau = \int_0^{\infty} [b+2c(t-\tau)] k^2(1-k\tau) e^{-k\tau} d\tau$$

$$\underline{= 2c ;}$$

that is, the time functions y_1 and y_2 will actually determine the sought-for derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.

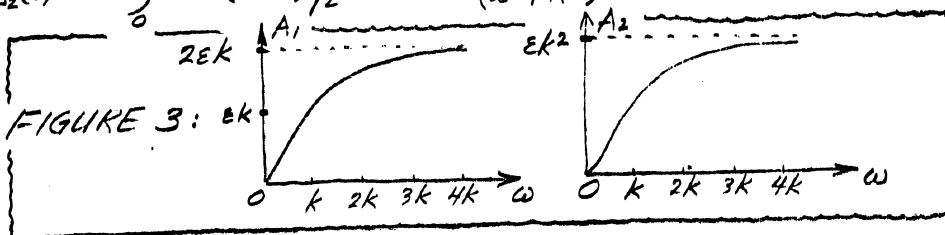
If the function $x(t)$ enters ("inputs") the differentiating unit of the apparatus with an error:

$$\delta x(t) = \varepsilon \sin \omega t \quad (106)$$

then in the determination of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ we will obtain for a steady state the errors $\Delta_1(t)$ and $\Delta_2(t)$, equal respectively to:

$$\Delta_1(t) = \varepsilon \omega \int_0^{\infty} \cos \omega(t-\tau) p_1(\tau) d\tau = \frac{\varepsilon \omega k}{(\omega^2 + k^2)^2} [k(3\omega^2 + k^2) \cos \omega t + 2\omega^3 \sin \omega t]$$

$$\Delta_2(t) = \varepsilon \omega \int_0^{\infty} \cos \omega(t-\tau) p_2(\tau) d\tau = \frac{\varepsilon \omega^2 k^2}{(\omega^2 + k^2)^2} [2\omega k \cos \omega t + (\omega^2 - k^2) \sin \omega t]$$



$$\text{or } \Delta_1(t) = A_1 \sin(\omega t + \phi_1) \quad \text{and} \quad \Delta_2(t) = A_2 \sin(\omega t + \phi_2) \quad (107)$$

$$\text{where } A_1 = \frac{\omega k \sqrt{4\omega^2 + k^2}}{\omega^2 + k^2} \varepsilon \quad \text{and} \quad A_2 = \frac{\omega^2 k^2}{\omega^2 + k^2} \varepsilon, \quad (108)$$

$$\phi_1 = \arctan \frac{k(3\omega^2 + k^2)}{2\omega^3} \quad \text{and} \quad \phi_2 = \arctan \frac{2\omega k}{\omega^2 - k^2}. \quad (109)$$

thus, if the error $\delta x(t)$ with which the function $x(t)$ enters ("inputs") the differentiating unit of the apparatus

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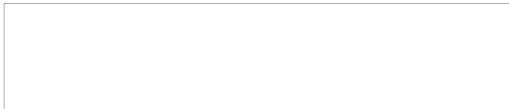
has the amplitude ϵ , then the errors of determination of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, namely, $\Delta_1(t)$ and $\Delta_2(t)$, will have the amplitudes A_1 and A_2 determined by the relations in (108). In Figure 3 are shown the graphs of the amplitudes A_1 and A_2 as functions of the frequency ω . As we see, no matter what the frequency ω may be, we always have:

$$|\Delta_1(t)| < 2\epsilon k \quad \text{and} \quad |\Delta_2(t)| < \epsilon k^2. \quad (110)$$

Substituting (100) into (89), we find that for any arbitrary character of the error $\delta x(t)$ we have:

$$|\Delta_1(t)| \leq 2\epsilon k \left(2 + \frac{1}{\epsilon^3}\right) \quad \text{and} \quad |\Delta_2(t)| \leq 2\epsilon k^2 \left(1 + \frac{1}{\epsilon^2}\right),$$

if $|\delta x(t)| \leq \epsilon$.



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